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Extended supersymmetry for the bound states of the generalized Hulthén potential hierarchy

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Abstract

Using the associated hypergeometric differential equation, we analytically solve the bound states corresponding to a hierarchy of the radial potential $-v_0\,\mathrm{e}^{-\delta r}/(1-\mathrm{e}^{-\delta r}) + c\,\mathrm{e}^{-\delta r}/(1-\mathrm{e}^{-\delta r})^2$ as a generalization of the Hulthén potential. Then, an analytic solution corresponding to a special case for which the parameter c is expected to be in terms of $l(l+1)$ is also derived. Meanwhile without introducing a superpotential and in the framework of supersymmetric quantum mechanics, it is shown that these bound states can be calculated by two different algebraic methods. Based on these two approaches, it is noted that the bound states realize an extended supersymmetry structure.

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1. Introduction and motivation

The factorization method was first introduced by Schrödinger to find exact solutions of non-relativistic quantum mechanics problems [1]. Then, this method was extended by Infeld and Hull, and others [2]. Afterwards, it was realized that [3] the factorization method of Schrödinger is a reconstruction of an old technique, the so-called Darboux transformations [4]. Following the pioneering ideas of Witten [5], it was understood that supersymmetry can be used in quantum mechanics as a limiting case of ($d = 1$) quantum field theory. Some authors soon found out that supersymmetric quantum mechanics is related to the factorization method of Schrödinger and Darboux transformations which, in turn, led to interesting applications of the theory in the fields of atomic, nuclear and condensed matter physics [6, 7]. In these references, it is shown that the supersymmetry approach to non-relativistic quantum mechanics provides not only a deep understanding of exactly solvable shape invariant Hamiltonians but also a powerful set of approximation schemes for dealing with problems that are not solved analytically. Gendenshtein obtained the relation between supersymmetry and solvable potentials by introducing the concept of shape invariant potentials [8]. According to his idea a potential is shape invariant if its supersymmetric partner potential has the same spatial dependence as the original potential so that only suitable parameters should be shifted in it. In fact, he showed that when two supersymmetric partner Hamiltonians realize the shape invariance relation then the corresponding wavefunctions and spectra can be calculated by an algebraic method. The concept of shape invariance seems to reach its highest power when it is coupled with supersymmetry. Over the last two decades, following these remarkable theories, much work has been done to investigate one-dimensional solvable quantum models in the framework of supersymmetric quantum mechanics and shape invariance [9–12]. It has been shown that shape invariance which appears as the Gendenshtein definition in connection with one-dimensional quantum mechanics problems, is a factorization method for solving differential equations [11]. Although for realization of the shape invariance, factorizability of a differential equation is not a sufficient condition, however, it is a necessary condition.

In order to describe the extended supersymmetry for the Hulthén problem, as well as its differences with ordinary supersymmetric quantum mechanics, it is necessary to explain briefly the well-known supersymmetry. It must be mentioned that, as shown in [12], the explanation of shape invariance in the framework of supersymmetry and in terms of parameters a_n and b_n (or α_n and β_n) is not different from explaining it in terms of n. In supersymmetric quantum mechanics, two factorized partner Hamiltonians $H_{n,+}(x) = A_n^\dagger(x)A_n(x)$ and $H_{n,-}(x) = A_n(x)A_n^\dagger(x)$ $(n = 0, 1, 2, \ldots)$ have the following eigenvalue equations:

$$\begin{aligned} H_{n,+}(x)\psi_n(x) &= \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_{n,+}(x)\right)\psi_n(x) = E_n\psi_n(x) \\ H_{n,-}(x)\psi_{n-1}(x) &= \left(-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V_{n,-}(x)\right)\psi_{n-1}(x) = E_n\psi_{n-1}(x), \end{aligned} \tag{1}$$

where

$$A_n(x) = -\frac{\mathrm{d}}{\mathrm{d}x} + W_n(x) \qquad A_n^\dagger(x) = \frac{\mathrm{d}}{\mathrm{d}x} + W_n(x), \tag{2}$$

$$V_{n,\pm}(x) = W_n^2(x) \pm \frac{\mathrm{d}W_n(x)}{\mathrm{d}x}. \tag{3}$$

$W_n(x)$ and $V_{n,\pm}(x)$ are superpotential and partner potentials, respectively (for a review see [9]). Introducing supercharges Q and $Q^\dagger$ as

$$Q = \begin{pmatrix} 0 & 0 \\ A_n(x) & 0 \end{pmatrix} \qquad Q^\dagger = \begin{pmatrix} 0 & A_n^\dagger(x) \\ 0 & 0 \end{pmatrix}, \tag{4}$$

the super-Hamiltonian H_{SUSY} which involves both of the components $H_{n,+}(x)$ and $H_{n,-}(x)$ takes the following form:

$$H_{\mathrm{SUSY}} = \{Q, Q^\dagger\} = \begin{pmatrix} H_{n,+}(x) & 0 \\ 0 & H_{n,-}(x) \end{pmatrix}. \tag{5}$$

It is shown that

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = [H_{\mathrm{SUSY}}, Q] = [H_{\mathrm{SUSY}}, Q^\dagger] = 0. \tag{6}$$

Relations (5) and (6) describe superalgebra $sl(1, 1)$ for two supercharge operators Q and $Q^\dagger$ and one bosonic operator H_{SUSY} [13]. In the context of unbroken supersymmetry one may obtain the following results. (a) If, from the relation

$$A_0(x)\psi_0(x) = 0, \tag{7}$$

the ground state $\psi_0(x)$ is derived as

$$\psi_0(x) = N_0 \exp\left(\int^x W_0(x')\,\mathrm{d}x'\right), \tag{8}$$

then the energy of this state is zero, i.e., $E_0 = 0$. (b) Equations (1) show that for a given n the Hamiltonians $H_{n,+}(x)$ and $H_{n,-}(x)$ have the same spectra for the bound states $\psi_n(x)$ and $\psi_{n-1}(x)$, respectively. (c) The eigenfunctions of the Hamiltonians $H_{n,+}(x)$ and $H_{n,-}(x)$ are related to each other as

$$A_n^\dagger(x)\psi_{n-1}(x) = \sqrt{E_n}\psi_n(x) \tag{9a}$$

$$A_n(x)\psi_n(x) = \sqrt{E_n}\psi_{n-1}(x). \tag{9b}$$

(d) Using the fact that the wavefunction of $H_{0,+}(x)$ is $\psi_0(x)$ (see equation (8)) and applying equation (9a), it is shown that the wavefunction of the nth state is computed by an algebraic method to yield

$$\psi_n(x) = \frac{A_n^\dagger(x)A_{n-1}^\dagger(x)\cdots A_1^\dagger(x)\psi_0(x)}{\sqrt{E_nE_{n-1}\cdots E_1}}. \tag{10}$$

In the above approach to supersymmetry, the existence of the superpotential is essential so that the partner potentials (3) and the wavefunctions (8) and (10) are calculated in terms of this superpotential. Obviously for $H_{n,-}(x)$, n begins from one so the ground state $\psi_0(x)$ is non-degenerate. The origin of the mentioned fact may be found in equation (8) which shows that the superpotential $W_0(x)$ is expressed as a logarithmic derivative of the ground state $\psi_0(x)$. In this paper for the generalized Hulthén potential, we extract an extended supersymmetry which has some differences with the above discussions as well as some similarities.

The Hulthén spherical potential [14], which is a special case of Eckart potential [15], possesses a short range and due to this fact it has applications in many areas of physics including nuclear and particle physics [16], atomic physics [17–19], solid state physics [20], etc. For example, as a recent application, the four-parameter form of the Hulthén potential for a diatomic molecule has attracted much attention [21]. The radial part of the Schrödinger equation in the presence of the Hulthén potential is

$$\frac{-\hbar^2}{2M}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\psi(r) + \left(\frac{-\alpha z\delta\,\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} + \frac{\hbar^2}{2M}\frac{l(l+1)}{r^2}\right)\psi(r) = E\psi(r), \tag{11}$$

where α, z and $\delta > 0$ are the fine-structure constant, atomic number and screening parameter determining the range for the Hulthén potential, respectively. Equation (11) is analytically solvable only for zero angular momentum (s wavefunction) [18, 19]. For $l \neq 0$, the calculation of bound-state energies for equation (11) is performed by perturbation expansion and variation methods [22, 23]. For the Hulthén potential, another version has been presented and investigated [24]. In one of the methods for solving equation (11), an effective approximation as $1/r^2 \simeq \delta^2\,\mathrm{e}^{-\delta r}/(1-\mathrm{e}^{-\delta r})^2$ is used for the centrifugal term in the case of $l > 0$ and small r. So, the second parenthesis in (11) is substituted by the following expression:

$$V_H^{\mathrm{eff}}(r) = \frac{-\alpha z\delta\,\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} + \frac{\hbar^2}{2M}\frac{l(l+1)\delta^2\,\mathrm{e}^{-\delta r}}{(1-\mathrm{e}^{-\delta r})^2}. \tag{12}$$

Using the above substitution and supersymmetric partner method in equation (11), p, 2p and d wave bounds for the Hulthén potential are derived [25]. Moreover, there properties of scattering solutions have also been examined. In [26], this examination has been done for the bound states by considering the related potential as a one-parameter trial function in a variational calculation of the screened Coulomb. Consequently, the energy levels of 2p, 3p, 3d, 4p, 4d and 4f states have been calculated. Considering an appropriate superpotential, in addition to applying the supersymmetry formalism for a potential similar to (12) ($e^{-\delta r}$ is replaced by $e^{-2\delta r}$ in the numerator of the second term) as discussed in [27, 28], perturbation and analytic expansion methods have been used for $l \neq 0$. It must also be mentioned that some other aspects of the Hulthén potential including the relations of phase equivalent and variational methods with supersymmetry have been investigated [29].

Regarding the basic ideas, [30] has attracted much attention in this paper. In the mentioned reference in order to obtain the normalization coefficients of wavefunctions corresponding to the shape invariance potentials (shape invariance as Gendenshtein's concept), a recursion relation between the normalization coefficients is constructed by using an operator method. Therefore, the necessity for introducing and applying the superpotential is emphasized in addition to keeping $N = 2$ ordinary supersymmetry structure as equations (1)–(10). One of the problems which has been discussed in [30] is the Hulthén potential. In this paper, on the basis of [31] and in the framework of supersymmetric quantum mechanics, we study the Hulthén problem from another point of view. In [31], mathematical aspects of shape invariance are examined, however, its physical applications are not studied. Secondly, the idea of [30] is followed by imposing recursion relations on the coefficients of associated hypergeometric functions (not the wavefunctions). Thirdly, in [31] for the first time, the idea of simultaneous shape invariance with respect to two parameters by four different ways is discussed. This extended concept of shape invariance constitutes the basis of this paper. Here, the radial Schrödinger equation for the extended Hulthén potential is considered when $l = 0$ as

$$\frac{-\hbar^2}{2M}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right)\psi(r) + \left(\frac{-v_0\,\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} + \frac{c\,\mathrm{e}^{-\delta r}}{(1-\mathrm{e}^{-\delta r})^2}\right)\psi(r) = E\psi(r). \quad (13)$$

The above equation may be known as an effective approximation for the Hulthén potential by assuming $c = l(l+1)\delta^2$. First of all, by a suitable method, equation (13) is compared with the associated hypergeometric differential equation (15) and consequently (13) is solved analytically. The coefficients v_0 and c, bound-state energies E and their corresponding wavefunctions $\psi(r)$ are calculated. In fact, using supersymmetry algebra for a hierarchy of the generalized Hulthén potential, we determine the potential parameters, i.e. v_0 and c, and the spectrum E so that the important property of supersymmetry based on converting the bound states to each other is revealed by the first-order differential operators (like equations (9*a*) and (9*b*)). It is shown that the associated hypergeometric functions impose four different types of laddering (shape invariance) relations on the wavefunctions of equation (13). This fact leads us to a hierarchy of solvable generalized Hulthén potentials which has some similarities with Gendenshtein's shape invariance [8] and supersymmetric quantum mechanics [5] from one side as well as differences with them from another side. In the hierarchy related to the known supersymmetric quantum mechanics, this shape invariance is realized with respect to only one parameter. Whereas in our new formalism, the shape invariance is established with respect to two parameters in four different ways. Meanwhile, these shape invariances hold from the operator point of view. In fact, by this new approach, we present a developed concept of supersymmetric quantum mechanics. It is also shown that for realization of solvability and supersymmetry structure, the existence of the superpotential described in equations (1)–(10) is not necessary.

2. Simultaneous realization of two different types of laddering equations by the associated hypergeometric functions

Before we consider the generalized Hulthén potential in detail we briefly recall some basic results concerning determination of the normalization coefficients of the associated hypergeometric functions such that they represent simultaneously two different types of laddering equations [31]. It has been shown that for all integers $n \geqslant 0$ and $0 \leqslant m \leqslant n$, the associated hypergeometric functions $F_{n,m}^{(\alpha,\beta)}(x)$ with the Rodrigues representation

$$F_{n,m}^{(\alpha,\beta)}(x) = \frac{a_{n,m}(\alpha,\beta)}{x^{\alpha+\frac{m}{2}}(1-x)^{\beta+\frac{m}{2}}}\left(\frac{\mathrm{d}}{\mathrm{d}x}\right)^{n-m}(x^{\alpha+n}(1-x)^{\beta+n}) \tag{14}$$

satisfy the following associated differential equation:

$$x(1-x)F_{n,m}^{\prime\prime(\alpha,\beta)}(x) + [\alpha+1-(\alpha+\beta+2)x]F_{n,m}^{\prime(\alpha,\beta)}(x) + \left[n(\alpha+\beta+n+1) + \frac{m[2(\alpha-\beta)x-(2\alpha+m)]}{4x(1-x)}\right]F_{n,m}^{(\alpha,\beta)}(x) = 0. \tag{15}$$

Moreover, for given real parameters $\alpha, \beta > -1$, it has been shown that the associated hypergeometric functions $F_{n,m}^{(\alpha,\beta)}(x)$ with $n \geqslant m$, for a given m, form an orthogonal set with respect to an inner product with the weight function $x^{\alpha}(1-x)^{\beta}$ in the interval $x \in (0, 1)$. The reader must not mistake this α for the fine-structure constant. Meanwhile, if the normalization coefficients $a_{n,m}(\alpha,\beta)$ are chosen as

$$a_{n,m}(\alpha,\beta) = (-1)^m\sqrt{\frac{\Gamma(\alpha+\beta+n+m+1)}{\Gamma(n-m+1)\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}}C(\alpha,\beta) \qquad n \geqslant m, \tag{16}$$

where $C(\alpha,\beta)$ is an arbitrary real constant independent of n and m, then it is found that

$$\int_0^1 F_{n,m}^{(\alpha,\beta)}(x)F_{n',m}^{(\alpha,\beta)}(x)x^{\alpha}(1-x)^{\beta}\,\mathrm{d}x = \delta_{nn'}h_n^2(\alpha,\beta) \tag{17}$$

where $h_n^2(\alpha,\beta)$, which is the square norm of the associated hypergeometric functions $F_{n,m}^{(\alpha,\beta)}(x)$, is given by

$$h_n^2(\alpha,\beta) = \frac{C^2(\alpha,\beta)}{\alpha+\beta+2n+1}. \tag{18}$$

The above result states that the norm of the associated hypergeometric functions $F_{n,m}^{(\alpha,\beta)}(x)$ is independent of the parameter m if we choose the normalization coefficient $a_{n,m}(\alpha,\beta)$ as relation (16). It should be emphasized that the associated hypergeometric functions $F_{n,m}^{(\alpha,\beta)}(x)$ and their differential equation (15) reduce to the hypergeometric polynomials and their corresponding differential equation when $m = 0$, respectively. Note that just choosing (16) for the normalization coefficients $a_{n,m}(\alpha,\beta)$ we can get simultaneous realization of laddering equations with respect to n and m.

The choice of (16) for the normalization coefficients allows us to separate the associated differential equation (15) as the raising and lowering equations of the index n, i.e.,

$$A_+(n,m;x)F_{n-1,m}^{(\alpha,\beta)}(x) = \sqrt{E(n,m)}F_{n,m}^{(\alpha,\beta)}(x) \tag{19a}$$

$$A_-(n,m;x)F_{n,m}^{(\alpha,\beta)}(x) = \sqrt{E(n,m)}F_{n-1,m}^{(\alpha,\beta)}(x), \tag{19b}$$

where

$$A_+(n,m;x) = x(1-x)\frac{\mathrm{d}}{\mathrm{d}x} - (\alpha+\beta+n)x + \frac{1}{2}(2\alpha+n) - \frac{(n-m)(\alpha-\beta)}{2(\alpha+\beta+2n)} \tag{20a}$$

$$A_-(n,m;x) = -x(1-x)\frac{\mathrm{d}}{\mathrm{d}x} - nx + \frac{n}{2} - \frac{(n-m)(\alpha-\beta)}{2(\alpha+\beta+2n)}, \tag{20b}$$

and

$$E(n,m) = \frac{(n-m)(\alpha+n)(\beta+n)(\alpha+\beta+n+m)}{(\alpha+\beta+2n)^2}. \tag{21}$$

Hence, the associated hypergeometric differential equation (15) can be factorized into products of first-order differential operators $A_+(n,m;x)$ and $A_-(n,m;x)$ as the shape invariance equations with respect to n. The indices + and − in the operators $A_+(n,m;x)$ and $A_-(n,m;x)$ denote the raising and lowering features of the index n, respectively.

Also, choosing the normalization coefficients $a_{n,m}(\alpha,\beta)$ as (16) one can covert the associated hypergeometric differential equation (15) as the following raising and lowering relations of the index m:

$$A_+(m;x)F^{(\alpha,\beta)}_{n,m-1}(x) = \sqrt{\mathcal{E}(n,m)}F^{(\alpha,\beta)}_{n,m}(x) \tag{22a}$$

$$A_-(m;x)F^{(\alpha,\beta)}_{n,m}(x) = \sqrt{\mathcal{E}(n,m)}F^{(\alpha,\beta)}_{n,m-1}(x), \tag{22b}$$

where

$$A_+(m;x) = \sqrt{x(1-x)}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{(m-1)(2x-1)}{2\sqrt{x(1-x)}} \tag{23a}$$

$$A_-(m;x) = -\sqrt{x(1-x)}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{2(\alpha+\beta+m)x-2\alpha-m}{2\sqrt{x(1-x)}}, \tag{23b}$$

and

$$\mathcal{E}(n,m) = (n-m+1)(\alpha+\beta+n+m). \tag{24}$$

The shape invariance of the differential equation (15) with respect to the parameter m is also realized by a factorization of it into products of first-order differential operators $A_+(m;x)$ and $A_-(m;x)$. Once again, the indices + and − in the operators $A_+(m;x)$ and $A_-(m;x)$ describe the raising and lowering role of the index m, respectively. Therefore, relation (16) plays an important role in representing the raising and lowering relations of the indices n and m simultaneously by means of the associated hypergeometric functions via equations (19a), (19b) and (22a), (22b).

3. Exact solutions for the bound states of the generalized Hulthén potential

Before studying supersymmetric aspects of the solutions, we use equations (14)–(18) and choose suitable values for v_0, c and energy spectrum E in the radial Schrödinger equation (13), then the bound states are calculated. The change of variable

$$x = 1 - \mathrm{e}^{-\delta r} \tag{25}$$

converts the interval $x \in (0,1)$ in equation (17) to the interval $r \in (0,\infty)$. Using the change of function

$$\psi(r(x)) = u(x)F^{(-2\gamma-1,2\eta+2\gamma)}_{n,m}(x) \qquad \text{with} \qquad u(x) = \frac{(1-x)^{\eta+\gamma}}{x^\gamma \ln(1-x)}, \tag{26}$$

as well as the change of variable (25) in equation (13), one can easily transform the differential equation (13) to a differential equation of type (15) for the associated hypergeometric functions $F^{(-2\gamma-1,2\eta+2\gamma)}_{n,m}(x)$. Regarding the mentioned fact, it appears that we should assume $\gamma < 0$ and

$\eta > -\gamma - \frac{1}{2}$. Comparing equation (13) with (15) and using the change of variable (25) and change of function (26), then choosing

$$v_0(\eta, \gamma; n, m) = \frac{\hbar^2\delta^2}{2M}[(\eta+n)^2 - 2m(\eta+\gamma) - (\eta+\gamma+m)^2] \tag{27}$$

and

$$c(\gamma; m) = \frac{\hbar^2\delta^2}{2M}[\gamma(\gamma+1) + m(m-4\gamma-2)], \tag{28}$$

it is seen that we may obtain the radial Schrödinger equation for the generalized Hulthén potential as

$$\left[\frac{-\hbar^2}{2M}\left(\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{2}{r}\frac{\mathrm{d}}{\mathrm{d}r}\right) + \frac{-v_0(\eta, \gamma; n, m)\,\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}} + \frac{c(\gamma; m)\,\mathrm{e}^{-\delta r}}{\left(1-\mathrm{e}^{-\delta r}\right)^2}\right]\left|\begin{matrix}\eta, \gamma\\ n, m\end{matrix}\right\rangle = E(\eta, \gamma; m)\left|\begin{matrix}\eta, \gamma\\ n, m\end{matrix}\right\rangle. \tag{29}$$

The bound states and energy spectrum can be calculated to yield

$$\left|\begin{matrix}\eta, \gamma\\ n, m\end{matrix}\right\rangle = N_n(\eta, \gamma)\frac{\mathrm{e}^{-(\eta+\gamma)\delta r}}{r(1-\mathrm{e}^{-\delta r})^{\gamma}}F_{n,m}^{(-2\gamma-1, 2\eta+2\gamma)}(1-\mathrm{e}^{-\delta r})$$
$$\text{with}\quad N_n(\eta, \gamma) = \frac{\sqrt{2(\eta+n)}}{C(-2\gamma-1, 2\eta+2\gamma)}, \tag{30}$$

$$E(\eta, \gamma; m) = \frac{-\hbar^2\delta^2}{2M}[(\eta+\gamma+m)^2 + 2m(\eta+\gamma)]. \tag{31}$$

Moreover by applying equation (17), it is noted that the set of bound states $\left|\begin{smallmatrix}\eta,\gamma\\ n,m\end{smallmatrix}\right\rangle$ with the same m but with different n constitutes an orthonormal set with respect to an inner product with measure $\frac{\delta r^2\,\mathrm{e}^{-\delta r}\mathrm{d}r}{1-\mathrm{e}^{-\delta r}}$, that is,

$$\left\langle\begin{matrix}\eta, \gamma\\ n, m\end{matrix}\right|\left.\begin{matrix}\eta, \gamma\\ n', m\end{matrix}\right\rangle = \int_{r=0}^{\infty}\left(N_n(\eta, \gamma)\frac{\mathrm{e}^{-(\eta+\gamma)\delta r}}{r(1-\mathrm{e}^{-\delta r})^{\gamma}}F_{n,m}^{(-2\gamma-1, 2\eta+2\gamma)}(1-\mathrm{e}^{-\delta r})\right)^{*}$$
$$\times\left(N_{n'}(\eta, \gamma)\frac{\mathrm{e}^{-(\eta+\gamma)\delta r}}{r(1-\mathrm{e}^{-\delta r})^{\gamma}}F_{n',m}^{(-2\gamma-1, 2\eta+2\gamma)}(1-\mathrm{e}^{-\delta r})\right)\frac{\delta r^2\,\mathrm{e}^{-\delta r}\,\mathrm{d}r}{1-\mathrm{e}^{-\delta r}} = \delta_{nn'}. \tag{32}$$

In fact, results (27) and (28) state that we deal with a family of the generalized Hulthén potentials as a hierarchy. In the next section we investigate supersymmetric properties with respect to a shift of n and m, and due to this fact we will assume that η and γ are constants. Relation (31) expresses that for a given m the energy spectrum has the same value for all of the orthogonal bound states. By means of fixing the constants $v_0(\eta, \gamma; n, m)$ and $c(\gamma; m)$, and determining the constants γ and η in terms of two parameters n and m, it is found that in a complicated method n and m describe one-dimensional quantization for the bound states of the generalized Hulthén potential.

As a special case, choosing $m = 0$ and by redefining parameters γ, η and n as

$$\begin{aligned} &l := -\gamma - 1 && l > -1 \\ &\zeta := \eta + \gamma && \zeta > -\tfrac{1}{2} \\ &p := n - \gamma && p > 0, \end{aligned} \tag{33}$$

the following results are obtained:

$$
\begin{aligned}
&v_0(\zeta+l+1,-l-1;\,p-l-1,0)=\frac{\hbar^2\delta^2}{2M}[(\zeta+p)^2-\zeta^2]\\
&c(-l-1;0)=\frac{\hbar^2}{2M}l(l+1)\delta^2\\
&E(\zeta+l+1,-l-1;0)=\frac{-\hbar^2\delta^2\zeta^2}{2M}\\
&\left|\begin{matrix}\zeta+l+1,-l-1\\ p-l-1,0\end{matrix}\right\rangle=N_{p-l-1}(\zeta+l+1,-l-1)\frac{\mathrm{e}^{-\zeta\delta r}(1-\mathrm{e}^{-\delta r})^{l+1}}{r}F^{(2l+1,2\zeta)}_{p-l-1,0}(1-\mathrm{e}^{-\delta r}).
\end{aligned}
\tag{34}
$$

By applying appropriate and new definitions for the parameters, solution (34) can be converted to an exact solution for the effective potential (12) with $l\neq 0$. Indeed, many authors have found the solution by using variational or perturbation expansion methods (for example, see [23, 26, 27]). For $l\neq 0$, the above analytic results may be compared with the results of [28]. Also, solution (34) can be compared with the solution given in [18].

4. Extended supersymmetry and the generalized Hulthén potential hierarchy

Now we can obtain four pairs of laddering relations for the bound states of generalized Hulthén potential. For this purpose, firstly, we define two pairs of laddering operators as

$$
\begin{aligned}
A_\pm(n,m;r)&:=[\delta u(x)A_\pm(n,m;x)u^{-1}(x)]_{x=1-\mathrm{e}^{-\delta r}}\\
&=\pm(1-\mathrm{e}^{-\delta r})\frac{\mathrm{d}}{\mathrm{d}r}+\left(\eta+n-\frac{1}{2}\mp\frac{1}{2}\right)\delta\,\mathrm{e}^{-\delta r}\pm\frac{1-\mathrm{e}^{-\delta r}}{r}\\
&\quad-\left(\eta+\gamma+\frac{n}{2}\right)\delta+\frac{(n-m)(2\eta+4\gamma+1)\delta}{2(2\eta+2n-1)},
\end{aligned}
\tag{35}
$$

$$
\begin{aligned}
A_\pm(m;r)&:=[\delta u(x)A_\pm(m;x)u^{-1}(x)]_{x=1-\mathrm{e}^{-\delta r}}\\
&=\pm\sqrt{\frac{1-\mathrm{e}^{-\delta r}}{\mathrm{e}^{-\delta r}}}\frac{\mathrm{d}}{\mathrm{d}r}\pm\left((\eta+\gamma)\delta+\frac{1}{r}\right)\sqrt{\frac{1-\mathrm{e}^{-\delta r}}{\mathrm{e}^{-\delta r}}}\pm\gamma\delta\sqrt{\frac{\mathrm{e}^{-\delta r}}{1-\mathrm{e}^{-\delta r}}}\\
&\quad+\frac{-4(\eta\mp\eta+m-1)\delta\,\mathrm{e}^{-\delta r}+(4\gamma\mp4\gamma+4\eta\mp4\eta+2m-1\mp1)\delta}{4\sqrt{(1-\mathrm{e}^{-\delta r})\,\mathrm{e}^{-\delta r}}},
\end{aligned}
\tag{36}
$$

where the explicit forms of them are calculated by using equations (20*a*), (20*b*), (23*a*) and (23*b*). Applying equations (19*a*), (19*b*), (22*a*) and (22*b*), one may derive the laddering relations for the bound states of generalized Hulthén potential with respect to the indices n and m, respectively, as

$$
A_+(n,m;r)\left|\begin{matrix}\eta,\gamma\\ n-1,m\end{matrix}\right\rangle=\frac{N_{n-1}(\eta,\gamma)}{N_n(\eta,\gamma)}\sqrt{E(\eta,\gamma;n,m)}\left|\begin{matrix}\eta,\gamma\\ n,m\end{matrix}\right\rangle \tag{37a}
$$

$$
A_-(n,m;r)\left|\begin{matrix}\eta,\gamma\\ n,m\end{matrix}\right\rangle=\frac{N_n(\eta,\gamma)}{N_{n-1}(\eta,\gamma)}\sqrt{E(\eta,\gamma;n,m)}\left|\begin{matrix}\eta,\gamma\\ n-1,m\end{matrix}\right\rangle, \tag{37b}
$$

$$
A_+(m;r)\left|\begin{matrix}\eta,\gamma\\ n,m-1\end{matrix}\right\rangle=\sqrt{\mathcal{E}(\eta;n,m)}\left|\begin{matrix}\eta,\gamma\\ n,m\end{matrix}\right\rangle \tag{38a}
$$

$$
A_-(m;r)\left|\begin{matrix}\eta,\gamma\\ n,m\end{matrix}\right\rangle=\sqrt{\mathcal{E}(\eta;n,m)}\left|\begin{matrix}\eta,\gamma\\ n,m-1\end{matrix}\right\rangle, \tag{38b}
$$

where

$$E(\eta, \gamma; n, m) = \frac{(n-m)(n-2\gamma-1)(2\eta+2\gamma+n)(2\eta+n+m-1)\delta^2}{(2\eta+2n-1)^2}, \tag{39}$$

$$\mathcal{E}(\eta; n, m) = (n-m+1)(2\eta+n+m-1)\delta^2. \tag{40}$$

As the operators given in (2) are Hermitian conjugates of each other with an inner product with measure x, the operators $A_+(m; r)$ and $A_-(m; r)$ are also Hermitian conjugates of each other with respect to the inner product given in (32); however, the operators $A_+(n, m; r)$ and $A_-(n, m; r)$ are not.

An important point is the fact that none of the pair operators $A_\pm(m; r)$ and $A_\pm(n, m; r)$ like the laddering operators (2) lead to the introduction of a superpotential which, like (3), gives the partner potentials. Thus, none of the expressions $E(\eta, \gamma; n, m)$ and $\mathcal{E}(\eta; n, m)$ is the spectrum for the potential corresponding to the radial Schrödinger equation. In fact, $E(\eta, \gamma; m)$ is the energy spectrum. Another significant point is the fact that the lack of superpotential does not mean that, like (8) and (10), we cannot calculate the bound states $\left|\begin{smallmatrix}\eta,\gamma\\ n,m\end{smallmatrix}\right\rangle$ by an algebraic method. Since, it is evident that by using equations (39) and (40) in (37*b*) and (38*a*), respectively, the following first-order differential equations are obtained:

$$A_-(m, m; r)\left|\begin{matrix}\eta, \gamma\\ m, m\end{matrix}\right\rangle = 0 \tag{41}$$

$$A_+(n+1; r)\left|\begin{matrix}\eta, \gamma\\ n, n\end{matrix}\right\rangle = 0. \tag{42}$$

Each of equations (41) and (42) is analogous with (7). One may easily deduce the solution of equation (41) as

$$\left|\begin{matrix}\eta, \gamma\\ m, m\end{matrix}\right\rangle = N_m(\eta, \gamma) a_{m,m}(-2\gamma-1, 2\eta+2\gamma) \frac{\mathrm{e}^{-(\eta+\gamma+\frac{m}{2})\delta r}}{r(1-\mathrm{e}^{-\delta r})^{\gamma-\frac{m}{2}}}, \tag{43}$$

which is consistent with the analytic solution (30). The solution of equation (42) is like (43) except that m is replaced by n. Now, we can write down the explicit forms of the operators that annihilate the bound states $\left|\begin{smallmatrix}\eta,\gamma\\ m,m\end{smallmatrix}\right\rangle$ and $\left|\begin{smallmatrix}\eta,\gamma\\ n,n\end{smallmatrix}\right\rangle$

$$A_-(m, m; r) = (1-\mathrm{e}^{-\delta r})\left[-\frac{\mathrm{d}}{\mathrm{d}r} + \frac{\mathrm{d}}{\mathrm{d}r}\ln\left|\begin{matrix}\eta, \gamma\\ m, m\end{matrix}\right\rangle\right] \tag{44}$$

$$A_+(n+1; r) = \sqrt{\frac{1-\mathrm{e}^{-\delta r}}{\mathrm{e}^{-\delta r}}}\left[\frac{\mathrm{d}}{\mathrm{d}r} - \frac{\mathrm{d}}{\mathrm{d}r}\ln\left|\begin{matrix}\eta, \gamma\\ n, n\end{matrix}\right\rangle\right]. \tag{45}$$

According to equations (41) and (42), we can say that the operators $A_-(m, m; r)$ and $A_+(n+1; r)$ annihilate the bound state $\left|\begin{smallmatrix}\eta,\gamma\\ n,m\end{smallmatrix}\right\rangle$ for $n = m$ and $m = n$, respectively. The existence of the coefficients $(1-\mathrm{e}^{-\delta r})$ and $\sqrt{\frac{1-\mathrm{e}^{-\delta r}}{\mathrm{e}^{-\delta r}}}$ does not allow us to speak explicitly about the superpotential in a customary concept. Despite the mentioned fact we may solve the problem algebraically, like (10), without solving a second-order differential equation. Since indeed, the bound states $\left|{\eta,\gamma \atop m,m}\right\rangle$ and $\left|{\eta,\gamma \atop n,n}\right\rangle$ are obtained by solving the first-order differential equations (41) and (42) as a lowest state for a given m and a highest state for a given n, respectively. Thus, each of equations (37a) and (38b) provides an algebraic solution for arbitrary bound states $\left|{\eta,\gamma \atop n,m}\right\rangle$ in terms of $\left|{\eta,\gamma \atop m,m}\right\rangle$ and $\left|{\eta,\gamma \atop n,n}\right\rangle$, respectively, as

$$\left|\begin{matrix}\eta,\gamma\\ n,m\end{matrix}\right\rangle = \frac{N_n(\eta,\gamma)}{N_m(\eta,\gamma)}\frac{A_+(n,m;r)A_+(n-1,m;r)\cdots A_+(m+1,m;r)\left|{\eta,\gamma \atop m,m}\right\rangle}{\sqrt{E(\eta,\gamma;n,m)E(\eta,\gamma;n-1,m)\cdots E(\eta,\gamma;m+1,m)}} \qquad n>m \tag{46}$$

$$\left|\begin{matrix}\eta,\gamma\\ n,m\end{matrix}\right\rangle = \frac{A_-(m+1;r)A_-(m+2;r)\cdots A_-(n;r)\left|{\eta,\gamma \atop n,n}\right\rangle}{\sqrt{\mathcal{E}(\eta;n,m+1)\mathcal{E}(\eta;n,m+2)\cdots\mathcal{E}(\eta;n,n)}} \qquad m<n. \tag{47}$$

So for the hierarchy of generalized Hulthén potential, we can calculate other bound states by using the bound state $\left|{\eta,\gamma \atop n,n}\right\rangle = \left|{\eta,\gamma \atop m,m}\right\rangle$ $(n=m)$ with two different methods. Note that combining equations (37a) and (37b) as well as (38a) and (38b), each of them by two different methods, gives the factorized equations with respect to the parameters n and m, respectively:

$$A_+(n,m;r)A_-(n,m;r)\left|\begin{matrix}\eta,\gamma\\ n,m\end{matrix}\right\rangle = E(\eta,\gamma;n,m)\left|\begin{matrix}\eta,\gamma\\ n,m\end{matrix}\right\rangle \tag{48a}$$

$$A_-(n,m;r)A_+(n,m;r)\left|\begin{matrix}\eta,\gamma\\ n-1,m\end{matrix}\right\rangle = E(\eta,\gamma;n,m)\left|\begin{matrix}\eta,\gamma\\ n-1,m\end{matrix}\right\rangle \tag{48b}$$

and

$$A_+(m;r)A_-(m;r)\left|\begin{matrix}\eta,\gamma\\ n,m\end{matrix}\right\rangle = \mathcal{E}(\eta;n,m)\left|\begin{matrix}\eta,\gamma\\ n,m\end{matrix}\right\rangle \tag{49a}$$

$$A_-(m;r)A_+(m;r)\left|\begin{matrix}\eta,\gamma\\ n,m-1\end{matrix}\right\rangle = \mathcal{E}(\eta;n,m)\left|\begin{matrix}\eta,\gamma\\ n,m-1\end{matrix}\right\rangle. \tag{49b}$$

Each of equations (48a), (48b), (49a) and (49b) is a copy of radial Schrödinger equation (29) so that after some manipulation, each of them converts to it. These equations are analogous with the Schrödinger equations (1) from the shape invariance point of view. Now it is clear that for given n and m, and using each of pair operators $A_\pm(n,m;r)$ and $A_\pm(m;r)$ one may construct two supercharges and one bosonic operator so that they satisfy commutation and anticommutation relations of superalgebra $sl(1,1)$ like (5) and (6).

Now we define two pairs of first-order differential laddering operators as

$$\begin{aligned} A_{+,-}(n,m;r) &:= \frac{1}{\delta}[A_-(m;r)A_+(n,m;r) - A_+(n,m-1;r)A_-(m;r)] \\ A_{-,+}(n,m;r) &:= \frac{1}{\delta}[A_-(n,m;r)A_+(m;r) - A_+(m;r)A_-(n,m-1;r)], \end{aligned} \tag{50}$$

$$\begin{aligned} A_{+,+}(n,m;r) &:= \frac{1}{\delta}[A_+(m;r)A_+(n,m-1;r) - A_+(n,m;r)A_+(m;r)] \\ A_{-,-}(n,m;r) &:= \frac{1}{\delta}[A_-(n,m-1;r)A_-(m;r) - A_-(m;r)A_-(n,m;r)], \end{aligned} \tag{51}$$

which have the following explicit forms (using (35) and (36)):

$$
\begin{aligned}
A_{\pm,\mp}(n, m; r) = {}& \pm\left(\frac{2\eta+2\gamma+n}{2\eta+2n-1} - \mathrm{e}^{-\delta r}\right)\sqrt{\frac{1-\mathrm{e}^{-\delta r}}{\mathrm{e}^{-\delta r}}}\frac{\mathrm{d}}{\mathrm{d}r} \\
& + \left(n - m + \frac{1}{2} \mp \frac{1}{2}\right)\delta\sqrt{(1-\mathrm{e}^{-\delta r})\,\mathrm{e}^{-\delta r}} + \left(\frac{2\eta+2\gamma+n}{2\eta+2n-1} - \mathrm{e}^{-\delta r}\right) \\
& \times \frac{4(\eta+m-1)\delta\,\mathrm{e}^{-\delta r} - (4\eta+4\gamma+2m-1\pm1)\delta \pm 4(1-\mathrm{e}^{-\delta r})/r}{4\sqrt{(1-\mathrm{e}^{-\delta r})\,\mathrm{e}^{-\delta r}}}
\end{aligned} \tag{52}
$$

$$
\begin{aligned}
A_{\pm,\pm}(n, m; r) = {}& \pm\left(\frac{2\gamma-n+1}{2\eta+2n-1} + \mathrm{e}^{-\delta r}\right)\sqrt{\frac{1-\mathrm{e}^{-\delta r}}{\mathrm{e}^{-\delta r}}}\frac{\mathrm{d}}{\mathrm{d}r} \\
& - \left(2\eta + n + m - 1 - \frac{1}{2} \mp \frac{1}{2}\right)\delta\sqrt{(1-\mathrm{e}^{-\delta r})\,\mathrm{e}^{-\delta r}} + \left(\frac{2\gamma-n+1}{2\eta+2n-1} + \mathrm{e}^{-\delta r}\right) \\
& \times \frac{-4(\eta+m-1)\delta\,\mathrm{e}^{-\delta r} + (4\eta+4\gamma+2m-1\mp1)\delta \pm 4(1-\mathrm{e}^{-\delta r})/r}{4\sqrt{(1-\mathrm{e}^{-\delta r})\,\mathrm{e}^{-\delta r}}}.
\end{aligned} \tag{53}
$$

Clearly, none of the pair operators $A_{\pm,\mp}(n, m; r)$ and $A_{\pm,\pm}(n, m, r)$ lead to the introduction of a superpotential for the generalized Hulthén potential. Using relations (37*a*), (37*b*), (38*a*) and (38*b*), we can obtain representations of the laddering relations by the bound states of the generalized Hulthén potential for the operators $A_{\pm,\mp}(n, m; r)$ and $A_{\pm,\pm}(n, m; r)$, respectively, as

$$
A_{+,-}(n, m; r)\left|\begin{matrix}\eta, \gamma \\ n-1, m\end{matrix}\right\rangle = \frac{N_{n-1}(\eta, \gamma)}{N_n(\eta, \gamma)}\sqrt{E_1(\eta, \gamma; n, m)}\left|\begin{matrix}\eta, \gamma \\ n, m-1\end{matrix}\right\rangle \tag{54a}
$$

$$
A_{-,+}(n, m; r)\left|\begin{matrix}\eta, \gamma \\ n, m-1\end{matrix}\right\rangle = \frac{N_n(\eta, \gamma)}{N_{n-1}(\eta, \gamma)}\sqrt{E_1(\eta, \gamma; n, m)}\left|\begin{matrix}\eta, \gamma \\ n-1, m\end{matrix}\right\rangle, \tag{54b}
$$

$$
A_{+,+}(n, m; r)\left|\begin{matrix}\eta, \gamma \\ n-1, m-1\end{matrix}\right\rangle = \frac{N_{n-1}(\eta, \gamma)}{N_n(\eta, \gamma)}\sqrt{E_2(\eta, \gamma; n, m)}\left|\begin{matrix}\eta, \gamma \\ n, m\end{matrix}\right\rangle \tag{55a}
$$

$$
A_{-,-}(n, m; r)\left|\begin{matrix}\eta, \gamma \\ n, m\end{matrix}\right\rangle = \frac{N_n(\eta, \gamma)}{N_{n-1}(\eta, \gamma)}\sqrt{E_2(\eta, \gamma; n, m)}\left|\begin{matrix}\eta, \gamma \\ n-1, m-1\end{matrix}\right\rangle, \tag{55b}
$$

where

$$
E_1(\eta, \gamma; n, m) = \frac{(n-m)(n-m+1)(n-2\gamma-1)(2\eta+2\gamma+n)\delta^2}{(2\eta+2n-1)^2}, \tag{56}
$$

$$
E_2(\eta, \gamma; n, m) = \frac{(2\eta+n+m-2)(n-2\gamma-1)(2\eta+2\gamma+n)(2\eta+n+m-1)\delta^2}{(2\eta+2n-1)^2}. \tag{57}
$$

Therefore, the operators $A_{+,-}(n, m; r)$ and $A_{-,+}(n, m; r)$ simultaneously increase one of the indices and decrease the other index. However, the operators $A_{+,+}(n, m; r)$ and $A_{-,-}(n, m; r)$ simultaneously increase and decrease both of the indices, respectively.

Now all of the necessary preliminaries for suggesting an extended supersymmetry realized by a hierarchy of the generalized Hulthén potential have been provided. For given n and m, supercharges $Q_i^{\pm}$ and bosonic operators H_i $(i = 1, 2, 3, 4)$ are defined as 8×8 matrices with elements:

$$
\begin{aligned}
&\left(Q_1^+\right)_{ij} = \delta_{i1}\delta_{j8}\frac{N_n(\eta,\gamma)}{N_{n-1}(\eta,\gamma)}A_+(n,m;r) \qquad\qquad (Q_1^-)_{ij} = \delta_{i8}\delta_{j1}\frac{N_{n-1}(\eta,\gamma)}{N_n(\eta,\gamma)}A_-(n,m;r)\\
&\left(Q_2^+\right)_{ij} = \delta_{i2}\delta_{j7}\sqrt{\frac{E(\eta,\gamma;n,m)}{\mathcal{E}(\eta;n,m)}}A_+(m;r) \qquad (Q_2^-)_{ij} = \delta_{i7}\delta_{j2}\sqrt{\frac{E(\eta,\gamma;n,m)}{\mathcal{E}(\eta;n,m)}}A_-(m;r)\\
&\left(Q_3^+\right)_{ij} = \delta_{i3}\delta_{j6}\frac{N_n(\eta,\gamma)}{N_{n-1}(\eta,\gamma)}\sqrt{\frac{E(\eta,\gamma;n,m)}{E_1(\eta,\gamma;n,m)}}A_{+,-}(n,m;r)\\
&(Q_3^-)_{ij} = \delta_{i6}\delta_{j3}\frac{N_{n-1}(\eta,\gamma)}{N_n(\eta,\gamma)}\sqrt{\frac{E(\eta,\gamma;n,m)}{E_1(\eta,\gamma;n,m)}}A_{-,+}(n,m;r)\\
&\left(Q_4^+\right)_{ij} = \delta_{i4}\delta_{j5}\frac{N_n(\eta,\gamma)}{N_{n-1}(\eta,\gamma)}\sqrt{\frac{E(\eta,\gamma;n,m)}{E_2(\eta,\gamma;n,m)}}A_{+,+}(n,m;r)\\
&(Q_4^-)_{ij} = \delta_{i5}\delta_{j4}\frac{N_{n-1}(\eta,\gamma)}{N_n(\eta,\gamma)}\sqrt{\frac{E(\eta,\gamma;n,m)}{E_2(\eta,\gamma;n,m)}}A_{-,-}(n,m;r)\\
&(H_1)_{ij} = \delta_{i1}\delta_{j1}A_+(n,m;r)A_-(n,m;r) + \delta_{i8}\delta_{j8}A_-(n,m;r)A_+(n,m;r)\\
&(H_2)_{ij} = \frac{E(\eta,\gamma;n,m)}{\mathcal{E}(\eta;n,m)}[\delta_{i2}\delta_{j2}A_+(m;r)A_-(m;r) + \delta_{i7}\delta_{j7}A_-(m;r)A_+(m;r)]\\
&(H_3)_{ij} = \frac{E(\eta,\gamma;n,m)}{E_1(\eta,\gamma;n,m)}[\delta_{i3}\delta_{j3}A_{+,-}(n,m;r)A_{-,+}(n,m;r)\\
&\qquad\quad + \delta_{i6}\delta_{j6}A_{-,+}(n,m;r)A_{+,-}(n,m;r)]\\
&(H_4)_{ij} = \frac{E(\eta,\gamma;n,m)}{E_2(\eta,\gamma;n,m)}[\delta_{i4}\delta_{j4}A_{+,+}(n,m;r)A_{-,-}(n,m;r)\\
&\qquad\quad + \delta_{i5}\delta_{j5}A_{-,-}(n,m;r)A_{+,+}(n,m;r)].
\end{aligned} \tag{58}
$$

The supercharges and bosonic operators satisfy the commutation and anticommutation relations of a superalgebra which, in turn, is a direct sum of superalgebras $sl(1,1)$ $(i, j = 1, 2, 3, 4)$ as follows:

$$
\begin{aligned}
&\left\{Q_i^+, Q_j^-\right\} = \delta_{ij}H_i\\
&\left\{Q_i^+, Q_j^+\right\} = \{Q_i^-, Q_j^-\} = 0\\
&\left[H_i, Q_j^\pm\right] = [H_i, H_j] = 0.
\end{aligned} \tag{59}
$$

The superstates $|i\rangle$ $(i = 1, 2, 3, 4)$ as 8×1 column matrices

$$
\begin{aligned}
&(|1\rangle)_j = \delta_{j1}\left|\begin{matrix}\eta,\gamma\\ n,m\end{matrix}\right\rangle + \delta_{j8}\left|\begin{matrix}\eta,\gamma\\ n-1,m\end{matrix}\right\rangle\\
&(|2\rangle)_j = \delta_{j2}\left|\begin{matrix}\eta,\gamma\\ n,m\end{matrix}\right\rangle + \delta_{j7}\left|\begin{matrix}\eta,\gamma\\ n,m-1\end{matrix}\right\rangle\\
&(|3\rangle)_j = \delta_{j3}\left|\begin{matrix}\eta,\gamma\\ n,m-1\end{matrix}\right\rangle + \delta_{j6}\left|\begin{matrix}\eta,\gamma\\ n-1,m\end{matrix}\right\rangle\\
&(|4\rangle)_j = \delta_{j4}\left|\begin{matrix}\eta,\gamma\\ n,m\end{matrix}\right\rangle + \delta_{j5}\left|\begin{matrix}\eta,\gamma\\ n-1,m-1\end{matrix}\right\rangle,
\end{aligned} \tag{60}
$$

represent the bosonic operators H_i as the following eigenvalue equations:

$$
H_i|j\rangle = \delta_{ij}E(\eta,\gamma;n,m)|j\rangle \qquad i, j = 1, 2, 3, 4. \tag{61}
$$

So, the bound states of the generalized Hulthén potential hierarchy as equation (29) lead to the realization of an extended supersymmetry algebra as equations (59).

The significance of the supersymmetry algebra (59) becomes obvious when we note that the representation of the bosonic operators H_i by the superstates $|j\rangle$ as (61) leads automatically to the realization of the four different types of factorized equations for the Schrödinger equation (29) in the framework of the shape invariance. It is clear that in equation (61) if $i = j$ and they take values 1, 2, 3 and 4, then we shall obtain four pairs of shape invariance equations (48a), (48b) and (49a), (49b) as well as

$$A_{+,-}(n,m;r)A_{-,+}(n,m;r)\begin{vmatrix}\eta,\gamma\\ n,m-1\end{matrix}\Big\rangle = E_1(\eta,\gamma;n,m)\begin{vmatrix}\eta,\gamma\\ n,m-1\end{matrix}\Big\rangle \tag{62a}$$

$$A_{-,+}(n,m;r)A_{+,-}(n,m;r)\begin{vmatrix}\eta,\gamma\\ n-1,m\end{matrix}\Big\rangle = E_1(\eta,\gamma;n,m)\begin{vmatrix}\eta,\gamma\\ n-1,m\end{matrix}\Big\rangle \tag{62b}$$

and

$$A_{+,+}(n,m;r)A_{-,-}(n,m;r)\begin{vmatrix}\eta,\gamma\\ n,m\end{matrix}\Big\rangle = E_2(\eta,\gamma;n,m)\begin{vmatrix}\eta,\gamma\\ n,m\end{matrix}\Big\rangle \tag{63a}$$

$$A_{-,-}(n,m;r)A_{+,+}(n,m;r)\begin{vmatrix}\eta,\gamma\\ n-1,m-1\end{matrix}\Big\rangle = E_2(\eta,\gamma;n,m)\begin{vmatrix}\eta,\gamma\\ n-1,m-1\end{matrix}\Big\rangle. \tag{63b}$$

respectively, so that all of them are converted to the Schrödinger equation (29) by some mathematical manipulations. Thus, the eigenvalue equations of the bosonic operators H_i represent a solution of the equation corresponding to a hierarchy of Hamiltonians of the generalized Hulthén potential. In fact, based on the mathematics involved in [31], the bound states corresponding to the hierarchy of the generalized Hulthén potential are solved by two algebraic methods (46) and (47), and these solutions convert to each other in four different ways, (37a), (37b), (38a), (38b), (54a), (54b) and (55a), (55b). Note that one may apply different factorization techniques introduced in [31] for all the problems that are solved by converting them to the associated hypergeometric differential equation. Moreover, the extended supersymmetry can be obtained for them. Regarding the type of problem which is solved, it is possible to find an appropriate physical interpretation. For example, in connection with the generalized Hulthén potential, according to the first type of supersymmetry when we shift the index n in the bound state $\big|{}^{\eta,\gamma}_{n,m}\big\rangle$ of (29) by the operators $A_{\pm}(n,m;r)$, then only v_0 changes. According to the second type of supersymmetry, when we shift the index m by the operators $A_{\pm}(m;r)$, then both the parameters c and v_0 change. However, simultaneous shift of n and m in two different ways gives rise to complicated changes of c and v_0. The following special case is considered. If we introduce the equations of lines parallel with the bisector of the first quadrant of the n–m plane as $n = m + d - 1$ where $d = 1, 2, 3, \ldots$ is the label of these lines, then it will be noted that the operators $A_{\pm,\pm}(m + d - 1, m; r)$ describe the shift of the bound states on the dth line. Therefore, considering the last type of supersymmetry, it appears that by choosing $\eta = d - 2\gamma - 1$ in the hierarchy related to the arbitrary dth line and for all of the bound states $\big|{}^{d-2\gamma-1,\gamma}_{m+d-1,m}\big\rangle$, the parameter c changes in terms of m as (28), however v_0 remains unchanged.

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